Tutorial 11 Remarks about Homework

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Homework 2

45. Laplace equations Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

Rek:

$$W_x = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x}$$

$$W_{xx} = \left(f_{uu} \frac{\partial u}{\partial x} + \underline{f_{uv} \frac{\partial v}{\partial x} \right) \frac{\partial u}{\partial x}} + f_u \frac{\partial^2 u}{\partial x^2} + \left(f_{vu} \frac{\partial u}{\partial x} + \underline{f_{vv} \frac{\partial v}{\partial x} \right) \frac{\partial v}{\partial x}} + f_v \frac{\partial^2 u}{\partial x^2}$$

Don't miss them

Homework 3

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \leq x \leq 1, 0 \leq y \leq 1$

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of* R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points of* R where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

Rek Don't miss step 2

of f appear somewhere in the lists made in Steps 1 and 2.

Rek Don't miss step 2.

$\nabla f = 0 \Rightarrow$ critical points

$(0,0)$ $\left(\frac{1}{2},\frac{1}{2}\right)$

Then look at the value of f on the boundary.

$f(0,y)$ $f(x,0)$ $f(1,y)$ $f(x,1)$

and find all the maximum or minmum of those function.

Don't just compute $f(0,0)$ $f(1,1)$ $f(0,1)$ and $f(1,0)$.

30. Let $f(x, y) = \frac{(x - y)}{(x + y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which

a. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest **b.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest

Rek You don't need to assume $\vec{u}=(a,b)$ $a^2+b^2=1$ and then find the maximum of $D_{u}f\left(-\frac{1}{2},\frac{3}{2}\right)$, you can just use the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Homework 5

7. Let $f = e^{2x} sin(3y)$ be defined on $\mathbb{R}^2$.
   a. Find the Taylor's polynomial $P_3(x,y)$ for f centred at $(0,0)$.
   b. Find the m such that $|f(x,y) - P_m(x,y)| \leq \frac{1}{100}$ for any (x,y) satisfying $x^2+y^2 < 4$. Explain your reason.

We can regard P_m as a approximation of f. Under this setting, one may ask the error bound estimate for $|f - P_m|$. This error bound can be controlled as follows: Let

$$E_n(x,y) := \frac{1}{n!} \max_{(x,y)\in\Omega} \{|\partial_x^n f|, |\partial_x^{(n-1)}\partial_y f|, ..., |\partial_y^n f|\}(|x-a_1|^2 + |y-a_2|^2)^{\frac{n}{2}}, \tag{1.5}$$

Then

$$|f - P_m|(x,y) \leq E_{m+1}(x,y). \tag{1.6}$$

Rek. $|\partial_y^{n-k}\partial_x^k f| \leq 2^k 3^{n-k} e^{2x} \leq 3^n e^{2x}$

when $x^2+y^2 \leq 4$ $x \leq 2$

$$\max_{x^2+y^2\leq 4}\{|\partial_x^n f|, |\partial_x^{(n-1)}\partial_y f|, \cdots, |\partial_y^n f|\} \leq 3^n e^4$$

$$E_{m+1} \leq \frac{1}{(m+1)!} 3^{m+1} e^4 \boxed{(x^2+y^2)^{\frac{m+1}{2}}} \leq \frac{1}{(m+1)!} 3^{m+1} e^4 2^{m+1}$$

Don't miss this term!

3. Let

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

and $(f(t), g(t)) = (t, 1)$. Solve the equation $X'(t) = AX(t) + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$ with the initial data $(x_0, y_0) = (1, -1)$

Rek: Look for M, such that $M^{-1}AM$ is a diagonal matrix

$$\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = 2 \quad \lambda_2 = -2$$

$$(\lambda_1 I - A)\, v_1 = 0 \Rightarrow v_1 = (1, 1)$$

$$(\lambda_2 I - A)\, v_2 = 0 \Rightarrow v_2 = (1, -1)$$

$$M = \begin{pmatrix} \underbrace{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}}_{v_1} & \underbrace{\begin{matrix} 1 \\ -1 \end{matrix}}_{v_2} \end{pmatrix} \qquad M^{-1}AM = \begin{pmatrix} \overset{\lambda_1}{\boxed{2}} & 0 \\ 0 & \underset{\lambda_2}{\boxed{-2}} \end{pmatrix}$$

If you write $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Then $M^{-1}AM = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $M^{-1} = \dfrac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If you want $M^{-1} = M^T$, you may chose

$$M = \begin{pmatrix} \underbrace{\boxed{\begin{matrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{matrix}}}_{\frac{v_1}{|v_1|}} & \underbrace{\boxed{\begin{matrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{matrix}}}_{\frac{v_2}{|v_2|}} \end{pmatrix} \Rightarrow M^{-1} = M^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\frac{v_1}{\|v_1\|}$ $\frac{v_2}{\|v_2\|}$